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NOTE

Bernstein Polynomials and Modulus of Continuity[1]

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This note describes several properties related to smoothness which are preserved by the operator given by Bernstein polynomials. © 2000 Academic Press

The Bernstein polynomials

$$B_n(f; x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) p_{n,j}(x), \qquad p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}$$

of a given function $f(x)$ on $[0, 1]$, besides the convergence and approximation, preserve some properties of the original function. For example:

(i) if $f(x)$ is non-decreasing, then for all $n \geqslant 1$, the $B_n(f; x)$ are non-decreasing;

(ii) if $f(x)$ is convex, then for all $n \geqslant 1$, the $B_n(f; x)$ are convex and

$$B_n(f; x) \geqslant B_{n+1}(f; x) \geqslant f(x), \qquad x \in [0, 1];$$

for other examples, cf. [7, Sect. 1.7; 5, Sect. 6.3]. Further studies on the convexity of the Bernstein polynomials can be found in [3, 4, 9]. Another property that the Bernstein polynomials preserve, proved by an elementary method in [2] (cf. [1] also), is that

(iii) if $f \in \mathrm{Lip}_A\, \mu$, then for all $n \geqslant 1$, $B_n(f; x) \in \mathrm{Lip}_A\, \mu$.

A function f belongs to the Lipschitz class $\mathrm{Lip}_A\, \mu$ where $0 < \mu \leqslant 1$ and $A \geqslant 0$ if $\omega(f; t) \leqslant At^\mu$ for $0 < t \leqslant 1$, where $\omega(f; t) = \max_{|x_2 - x_1| \leqslant t} |f(x_2) - f(x_1)|$ is the modulus of continuity of $f(x)$. The interesting and important

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thing of the fact in (iii) is that each of the Bernstein polynomials for $n \geqslant 1$ has the same Lipschitz order and the same Lipschitz constant as in the original function.

The aim of this note is to show a more general conclusion than (iii) and two new properties preserved by the Bernstein polynomials. A function $\omega(t)$ on $[0, 1]$ is called a modulus of continuity if $\omega(t)$ is continuous, non-decreasing, semi-additive, and $\lim_{t \to 0+} \omega(t) = \omega(0) = 0$. We denote the class of continuous functions on $[0, 1]$ satisfying the inequality $\omega(f; t) \leqslant \omega(t)$ by H^{ω}. We will prove that

(iii)* if $\omega(t)$ is a modulus of continuity, then $f \in H^{\omega}$ implies that for all $n \geqslant 1$, $B_n(f; x) \in H^{2\omega}$; if $\omega(t)$ is concave (upper convex), then $f \in H^{\omega}$ implies that for all $n \geqslant 1$, $B_n(f; x) \in H^{\omega}$;

(iv) if $\omega(t)$ is a modulus of continuity, then for each $n \geqslant 1$, $B_n(\omega; t)$ is also a modulus of continuity and $B_n(\omega; t) \leqslant 2\omega(t)$; if $\omega(t)$ is concave, then for each $n \geqslant 1$, $B_n(\omega; t)$ is a concave modulus of continuity and $B_n(\omega; t) \leqslant \omega(t)$;

(v) if $f(x)$ is a non-negative function such that $x^{-1}f(x)$ is non-increasing on $(0, 1]$, then for each $n \geqslant 1$, $x^{-1}B_n(f; x)$ is non-increasing also.

The conclusions in (iii)*, (iv), and (v) are closely connected, which can be seen from the following propositions:

(a) if $f(x)$ is concave on $[0, 1]$ and $f(0) = 0$, then $x^{-1}f(x)$ is non-increasing on $(0, 1]$;

(b) if $f(x)$ is a function such that $f(0) = 0$ and $x^{-1}f(x)$ is non-increasing on $(0, 1]$, then $f(x)$ is semi-additive, i.e., $f(x_1 + x_2) \leqslant f(x_1) + f(x_2)$, for $x_1, x_2, x_1 + x_2 \in [0, 1]$.

The proofs of (iii)*, (iv), and (v) are elementary and those of (iii)* and (iv) are only based on the following two representations of the Bernstein polynomial $B_n(f; x)$ derived in [2],

$$B_n(f; x_1) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} q_{n,k,l}(x_1, x_2)\, f\left(\frac{k}{n}\right), \tag{1}$$

$$B_n(f; x_2) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} q_{n,k,l}(x_1, x_2)\, f\left(\frac{k+l}{n}\right), \tag{2}$$

where $q_{n,k,l}(x_1, x_2) = (n!/(k!\, l!\, (n-k-l)!))\, x_1^k (x_2 - x_1)^l\, (1 - x_2)^{n-k-l}$.

Proof of (iv). For any modulus $\omega(t)$ of continuity and $n \geqslant 1$, the Bernstein polynomial $B_n(\omega; t)$ is continuous, non-decreasing, and satisfies

$\lim_{t \to 0} B_n(\omega, t) = B_n(\omega, 0) = \omega(0) = 0$. Making use of (1), (2), and the semi-additivity of $\omega(t)$, repeating the computation in [2, p. 198] gives that for $0 \leqslant t_1 < t_2 \leqslant 1$ and $t_1 + t_2 \leqslant 1$,

$$\begin{aligned} B_n(\omega; t_2) - B_n(\omega; t_1) &= \sum_{k=0}^{n} \sum_{l=0}^{n-k} q_{n,k,l}(t_1, t_2) \left(\omega\left(\frac{k+l}{n}\right) - \omega\left(\frac{k}{n}\right) \right) \\ &\leqslant \sum_{k=0}^{n} \sum_{l=0}^{n-k} q_{n,k,l}(t_1, t_2)\, \omega\left(\frac{l}{n}\right) \\ &= B_n(\omega; t_2 - t_1), \end{aligned}$$

which shows the semi-additivity of $B_n(\omega; t)$, so that $B_n(\omega; t)$ is a modulus of continuity.

If $\omega(t)$ is concave, it follows from (ii) that for each $n \geqslant 1$, $B_n(\omega; t)$ is concave and $B_n(\omega; t) \leqslant \omega(t)$. If $\omega(t)$ is not concave, then by [8, Theorem 3.2-3; 6, Lemma 7.1.5], there is a concave modulus $\omega^*(t)$ of continuity such that

$$\omega(t) \leqslant \omega^*(t) \leqslant 2\omega(t), \tag{3}$$

so that $B_n(\omega; t) \leqslant B_n(\omega^*; t) \leqslant \omega^*(t) \leqslant 2\omega(t)$. This finishes the proof of (iv).

Again repeating the computation in [2, p. 198] and applying (iv) prove (iii)*.

Proof of (v). Suppose $f(x)$ is a non-negative function such that $x^{-1}f(x)$ is non-increasing on $(0, 1]$. Direct computation gives that for $n \geqslant 1$,

$$\begin{aligned} \frac{d}{dx}\{x^{-1}B_n(f; x)\} &= \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \frac{d}{dx}\left\{\frac{p_{n,k}(x)}{x}\right\} + f(0)\frac{d}{dx}\left\{\frac{(1-x)^n}{x}\right\} \\ &= -\sum_{k=1}^{n-1} \left[\left(\frac{k}{n}\right)^{-1} f\left(\frac{k}{n}\right) - \left(\frac{k+1}{n}\right)^{-1} f\left(\frac{k+1}{n}\right) \right] \\ &\quad \times kx^{-1}p_{n-1,k}(x) - \frac{[1 + (n-1)x]\, f(0)}{x^2(1-x)^{1-n}}, \end{aligned}$$

which is non-positive by assumption. Hence $x^{-1}B_n(f; x)$ is non-increasing.

Remark. (1) Set $f(x) = x \log^2(e/2 + 1/x)$, $x \in (0, 1]$, and $f(0) = 0$. It is easy to find that $f(x)$ is increasing and $x^{-1}f(x)$ is decreasing, but $f''(x)$ changes in sign at $x_0 = 2/e \in (0, 1)$, which means that $f(x)$ does not remain concave on $[0, 1]$. This example shows that the property (v) is different from (ii).

(2) The following example indicates that for non-concave modulus of continuity, the inequality $B_n(\omega; t) \leqslant 2\omega(t)$ and that $f \in H^\omega$ implies $B_n(f; x) \in H^{2\omega}$ for all $n \geqslant 1$ cannot be improved.

For $n \geqslant 2$, we put

$$\omega_n(t) = \begin{cases} n^2 t, & 0 \leqslant t \leqslant n^{-2}; \\ 1, & n^{-2} \leqslant t \leqslant 1 - n^{-2}; \\ n^2(t-1)+2, & 1-n^{-2} \leqslant t \leqslant 1, \end{cases}$$

and $f_n(x) = \omega_n(x)$. It is obvious that $\omega_n(t)$ is a modulus of continuiy and $\omega(f_n; t) = \omega_n(t)$. For $n \geqslant 2$, we have $B_n(f_n; x) = B_n(\omega_n; x) = 1 + x^n - (1-x)^n$, so that $\lim_{n\to\infty} B_n(f_n; 1-n^{-2}) = 2$. It follows that for sufficiently large n, $B_n(f_n; 1-n^{-2}) > 2-\varepsilon$, consequently, $\omega(B_n(f_n); 1-n^{-2}) \geqslant B_n(f_n; 1-n^{-2}) - B_n(f_n; 0) = B_n(f_n; 1-n^{-2}) > (2-\varepsilon)\, f_n(1-n^{-2}) = (2-\varepsilon)\, \omega_n(f_n; 1-n^{-2})$.

(3) Finally, we propose the following problem. In (iii)* and (iv), restricting to the class of modulus of continuity such that the $t^{-1}\omega(t)$ are non-increasing, does there exist a number c, less than 2, such that $B_n(\omega; t) \leqslant c\omega(t)$ and $B_n(f; x) \in H^{c\omega}$ for all $n \geqslant 1$ provided $f \in H^{\omega}$? If it does, find out the smallest number c^*.

It should be noted that the inequalities in (3) cannot be improved, even for the class of modulus of continuity such that $t^{-1}\omega(t)$ are non-increasing.

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REFERENCES

1. W. R. Bloom and D. Elliott, The modulus of continuity of the remainder in the approximation of Lipschitz functions, *J. Approx. Theory* **31** (1981), 59–66.
2. B. M. Brown, D. Elliott, and D. F. Paget, Lipschitz constants for the Bernstein polynomials of a Lipschitz continuous function, *J. Approx. Theory* **49** (1987), 196–199.
3. G. Z. Chang and P. J. Davis, The convexity of Bernstein polynomials over triangles, *J. Approx. Theory* **40** (1984), 11–28.
4. G. Z. Chang and J. Z. Zhang, Converse theorems of convexity for Bernstein polynomials over triangles, *J. Approx. Theory* **61** (1990), 265–278.
5. P. J. Davis, "Interpolation and Approximation," Blaisdell, Waltham, MA, 1963.
6. N. P. Korneichuk, "Extremal Problems in Approximation Theory," Moscow, 1976. [In Russian]
7. G. G. Lorentz, "Bernstein Polynomials," Mathematical Expositions, Vol. 8, 1953.
8. Yongsheng Sun, "Approximation Theory of Functions," Vol. I, Beijing Normal Univ. Press, Beijing, 1989. [In Chinese]
9. Z. Ziegler, Linear approximation and generalized convexity, *J. Approx. Theory* **1** (1968), 420–433.